NOTES ON ARITHMETIC

FOR THE USE OF

THE CADETS

OF THE

Royal Military College of Canada.

BY

EDGAR KENSINGTON,

MAJOR ROYAL ARTILLERY.

PROFESSOR OF MATHEMATICS AND ARTILLERY.

Price, 40 Cents.

KINGSTON:

WILLIAM BAILIE, PRINTER.

1880.

PARIS
K35

NOTES ON ARITHMETIC

FOR THE USE OF

THE CADETS

OF THE

Royal Military College of Canada.

BY

EDGAR KENSINGTON,

MAJOR ROYAL ARTILLERY.

PROFESSOR OF MATHEMATICS AND ARTILLERY.

5 Arithmetic can

LIBRARY OF CONGRESS
A 5 2 4 4 4
FEB 10 1885
SMITHSONIAN DEPOSIT

Price, 40 Cents.

KINGSTON:
WILLIAM BAILIE, PRINTER.
1880.

W.M.S. Dec. 29, 1902

M. W. S. Oct. 29, 1902

NOTES.

(1) A number is divisible by 2 if the last figure is divisible by 2. **Simple properties of numbers.**

A number is divisible by 4 if the last two figures are divisible by 4.

A number is divisible by 8 if the last three figures are divisible by 8, and so on.

A number is divisible by 3 if the sum of the digits is divisible by 3.

A number is divisible by 9 if the sum of the digits is divisible by 9.

A number is divisible by 11 if the sum of the digits in the odd places equals the sum of the digits in the even places or differs by 11 or a multiple of 11.

(2) **Squares to 400.**

$13]^2$	$14]^2$	$15]^2$	$16]^2$	$17]^2$	$18]^2$	$19]^2$	$20]^2$
169	196	225	256	289	324	361	400

Note particularly the squares of 13, 17, 19.

Cubes to 1000

2^3	3^3	4^3	5^3	6^3	7^3	8^3	9^3
8	27	64	125	216	343	512	729

Powers of 2.

2, 4, 8, 16, 32, 64, 128, 256, 512, 1024.

Note particularly $2^5=32$; $2^{10}=1024$.

Powers of 3.

3, 9, 27, 81, 243, 729.

Powers of 5.

5, 25, 125, 625, 3125.

Powers of 7.

7, 49, 343.

(3) **Approximate roots.**

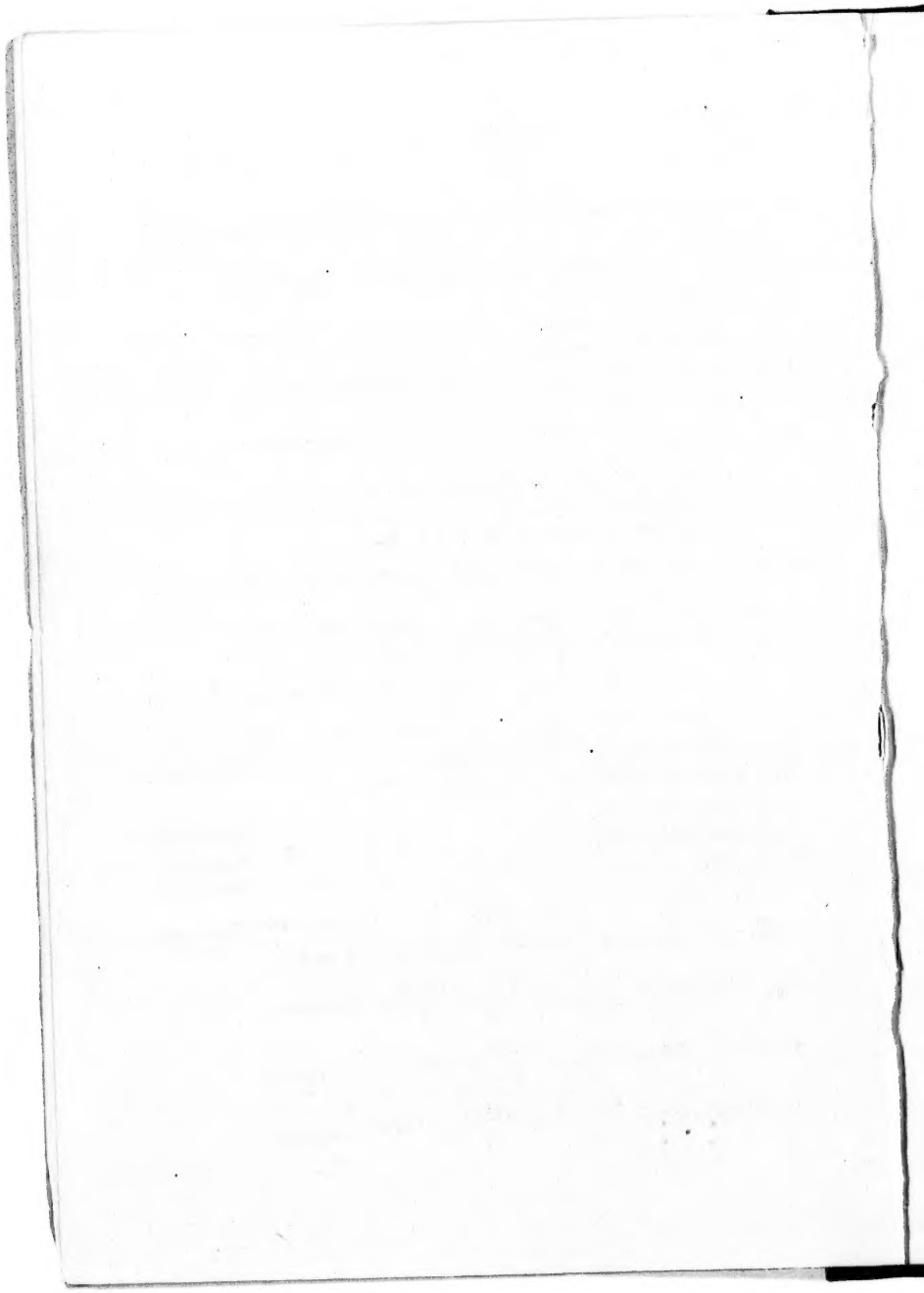
$\sqrt{2}=1{\cdot}414$ $\quad\sqrt{3}=1{\cdot}732$ $\quad\sqrt{5}=2{\cdot}236$

N.B. $\sqrt{1{\cdot}96}=1{\cdot}4$ $\quad\sqrt{2{\cdot}89}=1{\cdot}7$ $\quad\sqrt{4{\cdot}84}=2{\cdot}2$

Ex. (1) Sin. $45^\circ=\dfrac{1}{\sqrt{2}}=\dfrac{\sqrt{2}}{2}=\dfrac{1{\cdot}414}{2}={\cdot}707$ approx.

Ex. (2) Sin. $60^\circ=\dfrac{\sqrt{3}}{2}=\dfrac{1{\cdot}732}{2}={\cdot}866$ approx.

Ex. (3) Sin. $18^\circ=\dfrac{\sqrt{5}-1}{4}=\dfrac{1{\cdot}236}{4}={\cdot}309$ approx.



(4). It is generally best not to multiply out factors. Especially in the case of fractions reduced to a common denominator, the denominator should at first be left in factors. Numbers in factors.

(5). The simple rules of Arithmetic refer only to abstract numbers, no unit being given. An abstract number is always a ratio. Abstract numbers.

Ex. (1). 3 is the ratio $\frac{3}{1}$ just as much as $\frac{2}{3}$ is a ratio.

(6). A concrete number consists of two parts multiplied together, viz., a certain specified unit and an abstract number denoting how many times the unit is to be taken. Concrete numbers.

Ex. (1). The concrete number 3 horses = 3 × one horse. One horse is here the unit, 3 the abstract number.

Different numbers or ratios may express the same concrete number or quantity, provided different units are employed.

Ex. (2). Forty minutes = $40 \times 1^{\text{m.}} = \frac{2}{3} \times 1^{\text{hr.}}$

(7). *Concrete numbers, of like nature only, can be added or subtracted.* First, bring both to the same denomination; then add or subtract. Addition or subtraction of Concrete numbers.

Ex. (1). 3 horses cannot be added to 4 hours.
3 feet added to 4 inches = 40 inches.

(8). *Concrete numbers can be multiplied together in certain cases only;* namely, when the multiplication of the units gives an intelligible result. Multiplicat'n of Concrete numbers.

There are two most important elementary cases.

1st. Length × length = surface.

2nd. Length × length × length = volume; also, of course, surface × length = volume.

1st. Consider what is meant by multiplication of one inch by one inch.

An inch multiplied by an abstract number means that number of inches added to one another endways.

An inch multiplied by an inch means an inch added to itself not lengthways but broadways as many times as its

breadth is contained in its length. Thus the result is a square inch.

It may be objected that a line has no thickness, but this means no appreciable thickness, no thickness that can be reckoned on, in fact a thickness that is less than any that can be described or imagined.

To multiply length by length, the units must be multiplied and the numbers must be multiplied.

Ex. 3 inches $\times$ 4 inches = 12 square inches.
2 yards $\times$ 3 inches = 216 square inches.
$1\frac{1}{2}$ square feet.

Similarly one square inch multiplied by a linear inch, gives one cubic inch.

(9.) There are three elementary units on which all units employed in scientific researches may be made to depend, viz.: Units.

(1). The unit of length.
(2). The unit of time.
(3). The unit of mass.

Some of the other more important units are:

(4). The unit of velocity $= \frac{\text{space}}{\text{time}}$; $s = vt$.

(5). The unit of accelerating force $= \frac{\text{space}}{\text{time}]^2}$; $s = \frac{1}{2} ft^2$

(6). The unit of weight = mass $\times$ force; $W = mf$.

(7). The unit of momentum = mass $\times$ velocity; $M = m\,v$.

(8). The unit of dynamical energy = mass $\times$ vel]^2;

$$\text{Energy} = \frac{m\,v^2}{2} = \frac{w\,v^2}{2g}.$$

(9.) The unit of circular measure for measuring angles $= \frac{\text{arc}}{\text{radius}}$ = the ratio between two lengths.

Of these only No. 4 can be considered here.

Ex. A man walks at 3 miles per hour for 4 hours; he walks 12 miles, obtained by multiplying the velocity by the time.

In such examples care must be taken to use the same units of length or time throughout; or, if it be necessary to alter them, to perform the necessary arithmetical operation.

Ex. A locomotive goes 44 feet per second, how many miles does it go in an hour?

Ans. $\frac{44}{5280} \times 3600 = 30$ miles.

N.B. 30 miles per hour = 44 feet per second.

Here $\frac{44}{5280} = v$: i.e. the No. of miles per second.

$3600 = t$: i.e. the No. of seconds the journey lasted.

(10). It is evident that *corresponding to every case of multiplication there is a converse case of division.* Division of Concrete numbers.

Ex. (1). $\frac{\text{surface}}{\text{length}} = \text{length}$; $\frac{\text{space}}{\text{time}} = \text{velocity}$.

N.B. It will be found a safe rule to put the symbol of division for the word "per."

Ex. Velocity is 3 miles per hour $= \frac{3 \text{ miles}}{1 \text{ hour}} = \frac{4{\cdot}4 \text{ feet}}{1 \text{ sec'd}}$

Price is \$3.60 per yard $= \frac{\$3.60}{1 \text{ yard}} = \frac{\$1.20}{1 \text{ foot}} = \frac{10 \text{ cts.}}{1 \text{ inch}}$

Rate of interest is 5 per cent $= \frac{5}{100} = \frac{1}{20}$

Concrete numbers of like nature can be divided, the quotient being an abstract number or ratio.

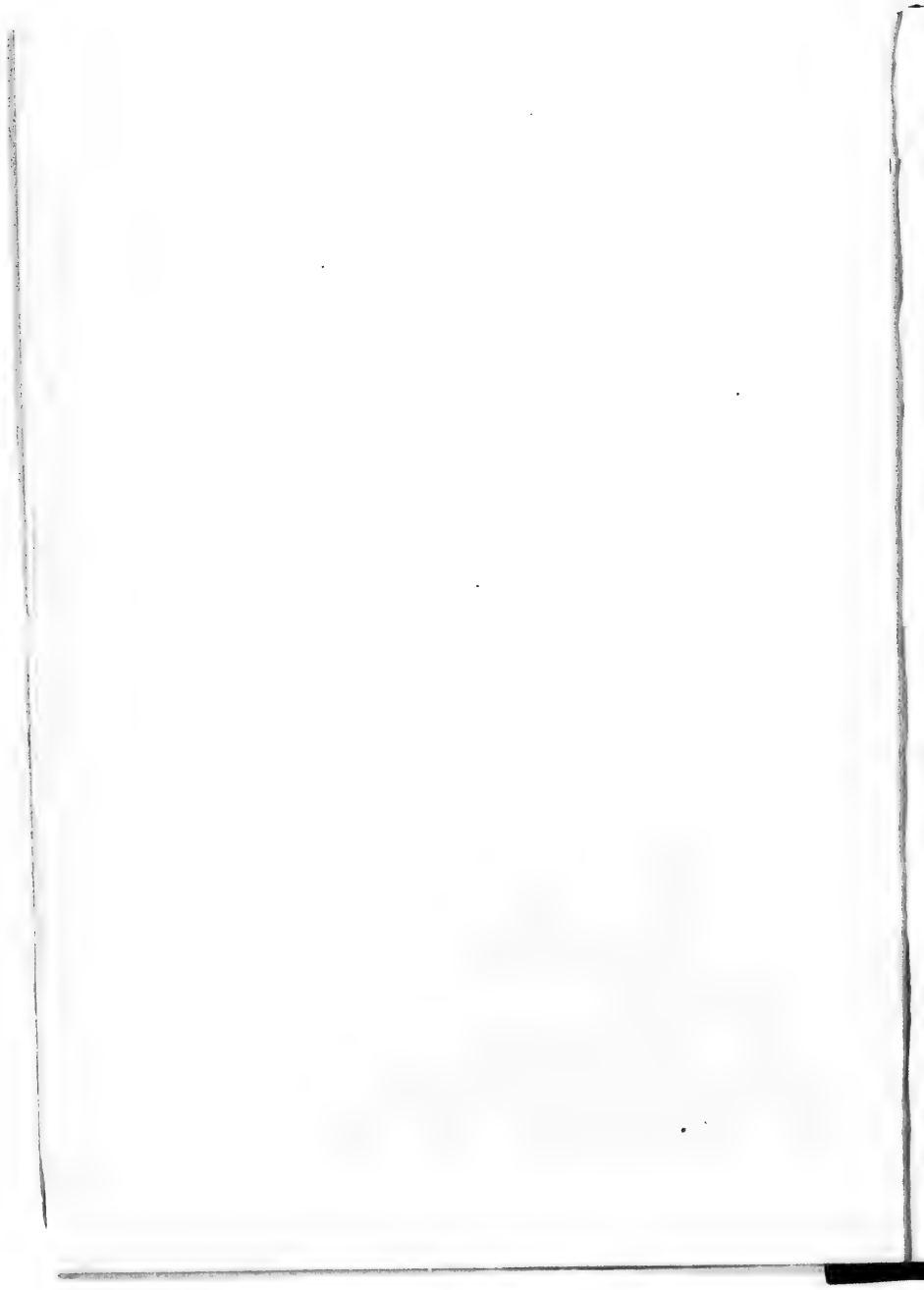
Ex. (2). How many times will 2ft. 3 in. go into 3 yds.?

$\frac{3 \text{ yards}}{2 \text{ ft. } 3 \text{ in.}} = \frac{9 \times 1 \text{ ft.}}{2\frac{1}{4} \times 1 \text{ ft.}} = \frac{9}{2\frac{1}{4}} = 4$, i.e. an abstract number, because the concrete unit of length has been divided out.

Ex. (3). How many times will a gallon go into a cubic foot?

Given $\left\{\begin{array}{l}\text{a cubic ft. of water weighs 1000 ozs.}\\ \text{a gallon of water weighs 10 lbs.}\end{array}\right\}$ nearly

$\frac{1 \text{ cubic ft.}}{1 \text{ gallon}} = \frac{1000 \times (\text{what holds 1 oz. of water})}{16 \times 10 \times (\text{what holds 1 oz. of water})} = \frac{1000}{160} = 6{\cdot}25.$



Here also the unit of capacity divides out, leaving an abstract number for the quotient.

(11). A correct appreciation of units and dimensions is of the greatest importance in Mathematics, and forms one of the simplest aids to memory and a frequent means for detecting errors in work.

See Notes on Algebra, page 3, sec 10.

Rules for Concrete numbers summarised.

(12). *A concrete number = the abstract number × the unit. Only concrete numbers of like nature can be added or subtracted.*

Concrete numbers cannot be multiplied together except in certain cases, of which the simplest are

Length × length = surface.
Surface × length = volume.
Velocity × time = distance travelled; $s = vt$.

Concrete numbers cannot be divided except in the corresponding cases, and *when both are of like nature, in which case the quotient is an abstract number or ratio.*

Weights.

(13). Troy weight, 1 lb. = 12 ozs. = 12×20×24 grains = 5760 grains; only used for precious metals and special purposes.

Apothecaries' weight, 1 lb. = 12 ozs. = 12×8×3×20 grs. = 5760 grains; only used for drugs and chemicals.

Note the similarity to Troy weight.

Avoirdupois weight, 1 lb. = 16 ozs. = 7000 grains Troy; used for all general purposes.

Note that the grain is the only connection between this and the former. The pound and ounce are both different.

N.B.—In England 1 cwt. = 4 × 28 lbs. = 112 lbs.
In Canada 1 cwt. = 4 × 25 lbs. = 100 lbs.

Length and surface.

(14). The clue to the measures of length is this:
22 yards = 1 chain = 100 links.
4840 sq. yds. = 1 acre = 10 × [22 yards]^2 = 10 square chains.

Then it will be seen that

1 pole = $5\frac{1}{2}$ yards = $\frac{1}{4}$ chain = 25 links.
1 furlong = 220 yards = 10 chains.
1 mile = 8 furlongs = 80 chains.

NOTE. Chains, poles, links; cwts., quarters, pounds; dollars, quarters, cents; } { exactly correspond, and have the advantage of being decimal measures.

A ton would correspond to a twenty dollar bill.

A mile would correspond to a bill for eighty dollars.

Ex. (1). Reduce 3 tons, 17 cwt., 3 qrs., 9 lbs. to hundredweights.

$$\underbrace{\qquad\qquad}_{84}$$

Ans. 77·84 cwt. = 7784 lbs.

Ex. (2). Reduce 3 miles, 5 fur., 37 poles, $2\frac{3}{4}$ yds. to chains.

240 chains + 50 chains + $9\frac{1}{4}$ chains + $\frac{11}{4\times22}$ chains.

Ans. $299\frac{1}{4} + \frac{1}{8} = 299\frac{3}{8}$ chains = 299·375 chains.

The great use of measurement by chain is that 1 acre=10 square chains.

Ex. (3). Reduce 37|2·5783 square chains to acres, roods, &c.

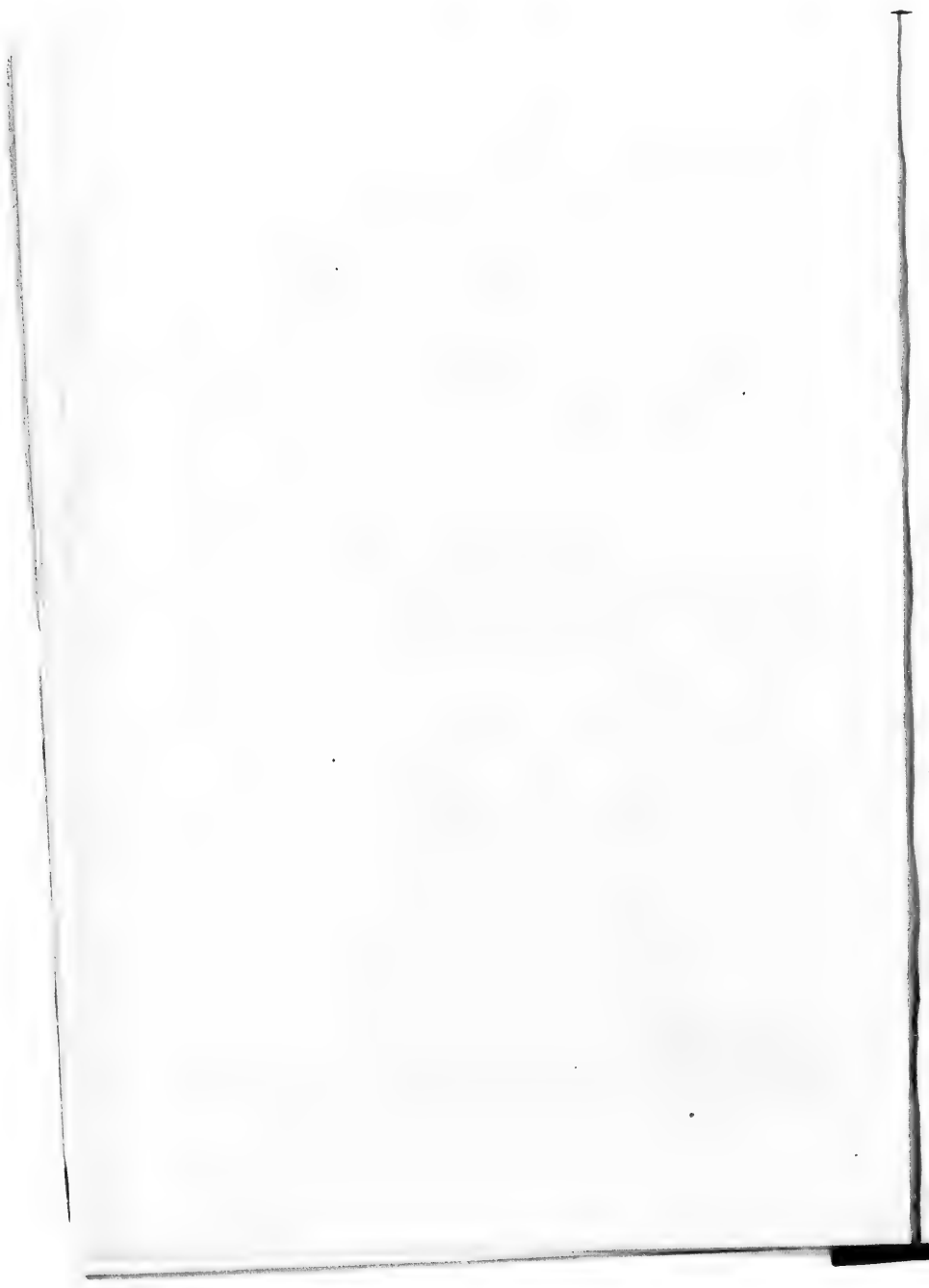
		4
roods	1	0·3132
		40
poles	1	2528 × 121/4
		632
		11
		6952
		11
yards	7	6472
		9
feet	5	8248
		12
	9	9376
		12
inches	119	2512

or

1	2528
	30½
7	5840
	632
	6472
	9

This reads as follows:

The vertical line serves to mark the position of the decimal point after dividing by 10.



372·5783 sq. chains=37·25783 acres.

acres	roods.
=37	1·03132

acres	rood	poles.
=37	1	1·2528

acres	rood	pole	sq. yds.
=37	1	1	7·6472

acres	rood	pole	sq. yds.	sq. ft.
=37	1	1	7	5·8248

acres	rood	pole	sq. yds.	sq. ft.	sq. in.
=37	1	1	7	5	119·2512

(15). Break up the multiplier into simple factors, if possible, and multiply by them in the most convenient order. Compound Multiplicat'n

Ex. (1). Multiply 2 miles, 2 fur., 13 poles, $3\frac{1}{2}$ yds. by 308.

$308\frac{1}{2} = 4 \times 7 \times 11$.

Multiply first by 11 to get rid of yards and fractions, for 11 half yards = 1 pole $\therefore$ 11 × $3\frac{1}{2}$ yards = 7 poles.

Multiply next by 4 to get rid of poles.

m.	*f.*	*p.*	*yds.*
2	2	13	$3\frac{1}{2}$
			11
25	1	30	
		4	
100	7		

miles 706 : 1 f. Ans.

Much time, trouble and consequent risk of error may be saved by a little consideration of this kind.

Secondly, if the multiplier cannot be broken into simple factors, the best way is generally as follows:

Multiply by as many 10s as necessary, and each line by the corresponding unit.

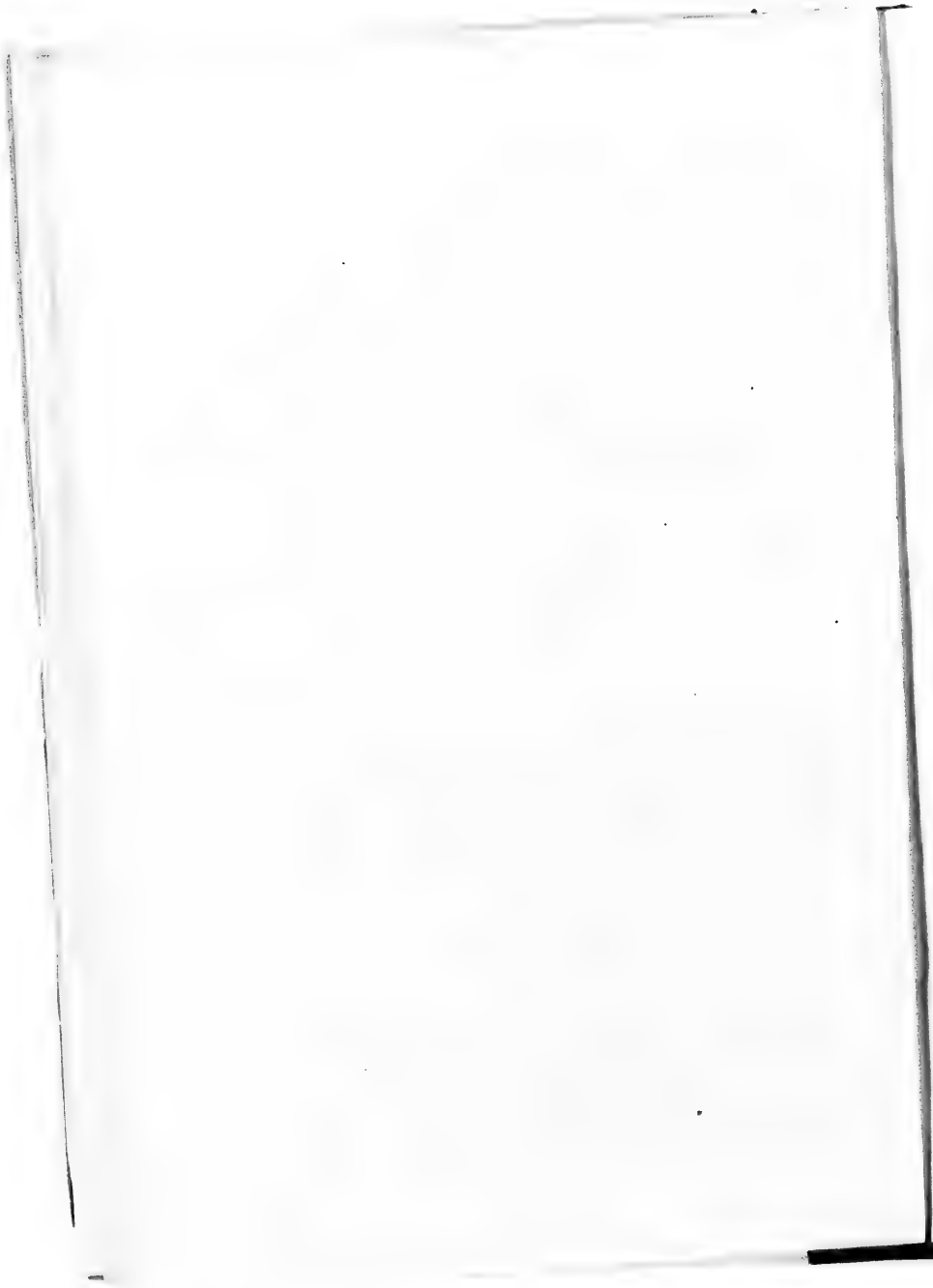
Ex. (2). Multiply

£	*s.*	*d.*
2	13	$3\frac{1}{2}$

by 239.

	£	*s.*	*d.*	
	26	12	11	
	266	9	2	
			2	
	532	18	4	= 200 × 2 : 13 : $3\frac{1}{2}$
3 × 2nd line =	79	18	9	= 30 × do.
9 × 1st line =	23	19	$7\frac{1}{2}$	= 9 × do.
	636	16	$8\frac{1}{2}$	

In special cases other methods of breaking up the multiplier may be found more convenient.



(16). No such process as the latter can be adopted for division; but it is generally best to break up the divisor into factors, when possible, so as to work by simple division instead of by long division. Compound Division.

Ex. Divide £2 : 13 : $3\frac{1}{2}$ by 308; $308 = 4 \times 7 \times 11$

4) 2 : 13 : $3\frac{1}{2}$

7) 13 : $3\frac{7}{8}$ for the remainder $3\frac{1}{2}$ divided by 4 is $\frac{7}{8}$.

11) 1 : $10\frac{47}{56}$ for $5\frac{7}{8}$ i.e. $\frac{47}{8}$ remains to be divided by 7

$2\frac{47}{616}$

The fractions must at each step be left in their lowest terms.

If any of the fractions is improper, i.e. > 1, it is a sure sign that there is a mistake.

(17). It is usual, in reducing fractions to their lowest terms, to *divide out any simple factor* of both numerator and denominator that can be observed, before resorting to the lengthy process of finding the G. C. M. G. C. M.

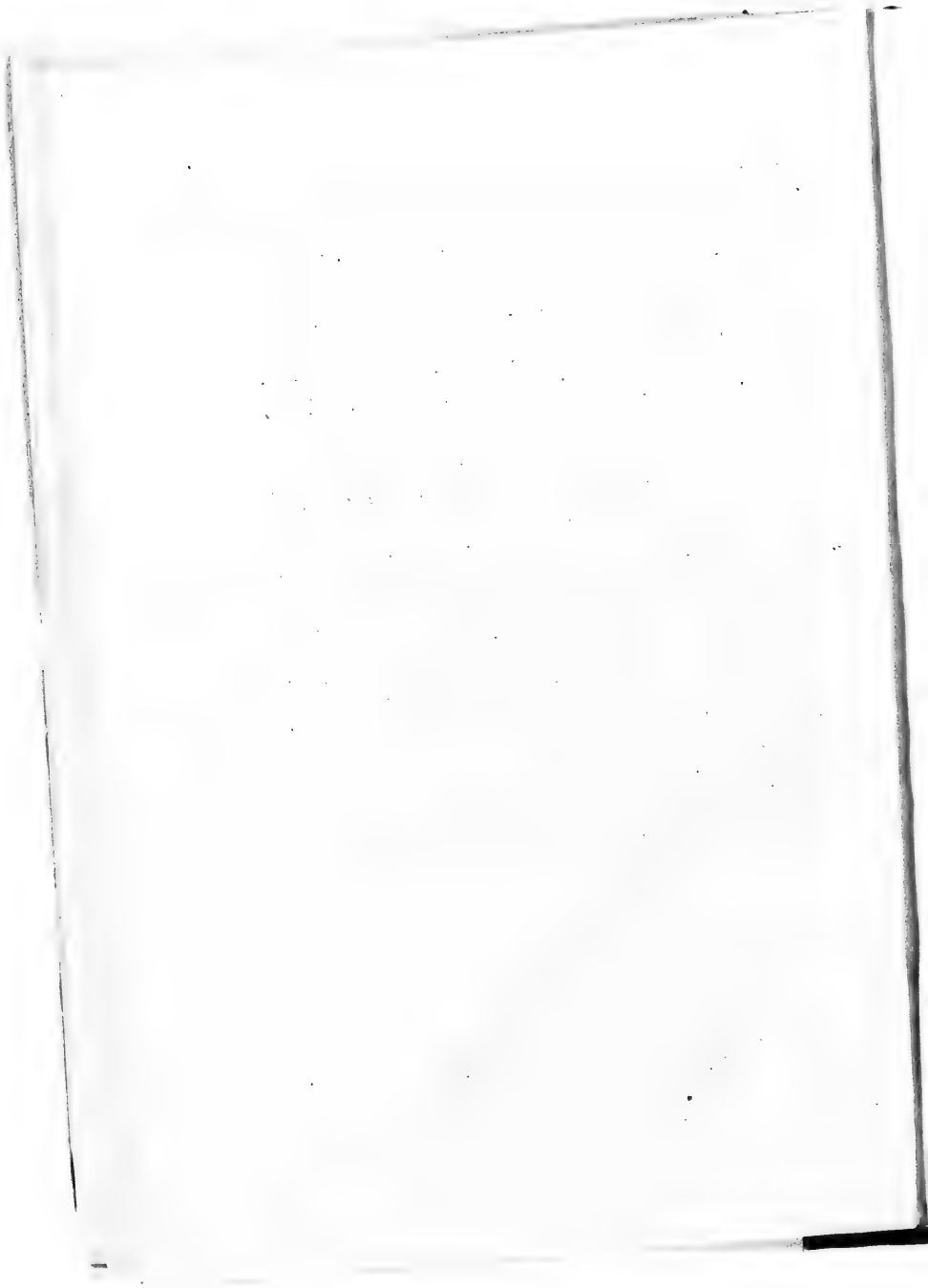
In like manner a question in G. C. M. can generally be much shortened by observing such simple factors.

Rule. *Strike out simple factors at any time* during the process, *retaining those only which divide both divisor and dividend.*

These latter fractions multiplied by the final result give the required G. C. M.

Ex. (1). Worked at length by the ordinary process.

```
10395)16819(1
      10395
       6424)10395(1
            6424
            3971)6424(1
                 3971
                 2453)3971(1
                      2453
                      1518)2453(1
                           1518
                            935)1518(1
                                935
                                583)935(1
                                   583
                                   352)583(1
                                      352
                                      231)352(1
                                         231
                                         121)231(1
                                            121
                                            110
```



It is unnecessary to work further, for the result will evidently be 11, because 11 is the G.C.M. of $\left\{\begin{matrix}121\\110\end{matrix}\right\}$

It is shewn in "Notes on Algebra", p. 15, that this G.C.M. is the G. C. M. of every pair, viz.: $\left\{\begin{matrix}10395\\16819\end{matrix}\right\}$; $\left\{\begin{matrix}6424\\10395\end{matrix}\right\}$; $\left\{\begin{matrix}3971\\6424\end{matrix}\right\}$ &c., &c., $\left\{\begin{matrix}352\\583\end{matrix}\right\}$; $\left\{\begin{matrix}231\\352\end{matrix}\right\}$; $\left\{\begin{matrix}121\\231\end{matrix}\right\}$

If then at any time during the process, the G.C.M of any pair can be seen or obtained by a direct method, this is the required G. C. M.

For instance $\left\{\begin{matrix}352 = 11 \times 32\\583 = 11 \times 53\end{matrix}\right\}$; this is easy to see; now 32 and 53 are prime to one another, therefore 11 must be the G. C. M.

Same Example worked by the above rule.

5)10395 | 16819, for there is no 5 in 16819.
9)2079 | for there is no 9 or 3 or 16819.
3)231 |
$77 = 7 \times 11$ |

11 is the only factor of 10395 that divides 16819. Therefore 11 is the G. C. M.

Ex. (2). 110)1335510 | 8)115872 — 2 divides both.
3)12141 | 14484 — 3 "
3)4047 | 4)4828
1207)1349 | 1207(17
1207 |
2)142 |
71 which divides 1207 exactly.

$\therefore$ G. C. M. $= 2 \times 3 \times 71$.

Here by striking out simple factors, the original numbers are reduced to $\left\{\begin{matrix}1349\\1207\end{matrix}\right\}$ whose G. C. M. is 71.

Observe that the next pair is $\left\{\begin{matrix}142\\1207\end{matrix}\right\}$ which is simplified again by striking out the factor 2, which is not contained in 1207.

Ex. (3). Find the G. C. M. of 805, 1311, 1978.

Strike out every factor which is not contained in every one. Thus 805 reduces to 23 a prime number; if then there is any G. C. M. it can be no other than 23.

5)805	3)1311	2)1978
7)161	437(19	989(43
23	23	92
	207	69
	207	69 Ans. 23.

It should be noticed, without dividing, that 23 divides 437 and 989.

The rule given above is intended to supplement, not to supersede, the ordinary rule.

Lengthy processes should always be avoided, if possible, because they lead to error.

(18). In the ordinary process the rule is to divide by prime numbers only. L. C. M.

The rule may be shortened if composite divisors are used with care.

Rule. When using composite divisors, *divide each number by as many of the factors of the divisor as possible.*

Rule. Shorten the process at any time by crossing out any number which is contained by another number in the same row. (Shewn below in block type).

Ex. 10)	72,	30,	**36,**	220,	275,	99, 162
12)	36,	**3,**		22,	55,	99, 81
33)	**3,**			**11,**	55,	33, 27
				5,		9

Ans. $10 \times 12 \times 33 \times 5 \times 9 = 600 \times 297 = 178200$

The result is the same as the following:

5)	72,	30,	**36,**	220,	275,	99, 162
2)	72,	**6,**		44,	55,	99, 162
3)	36,			22,	55,	99, 81
2)	12,			22,	55,	33, 27
2)	6,			**11,**	55,	33, 27
11)	**3,**				55,	33, 27
				5,	**3,**	27

There is a far shorter and more practically useful method of ascertaining the L. C. M. in factors, at sight.

Consider the highest power of each of the prime factors of all the given numbers.

Thus, in the above example,

Highest power of 2 is 8 contained in 72
" " 3 " 81 " 162
" " 5 " 25 " 275
" " 11 " 11 " 275 and others.

$\therefore$ L.C.M. $= 8 \times 81 \times 25 \times 11 = 200 \times 891 = 178200$.

The least common multiple is generally required for the addition of fractions.

(19). If the above numbers represent the denominators of several fractions, multiply the numerators as follows: Addition of Fractions.

That over 72 i.e. 8×9 by $9 \times 25 \times 11$
" 30 " $2 \times 3 \times 5$ " $4 \times 27 \times 5 \times 11$
" 36 " 4×9 " $2 \times 9 \times 25 \times 11$
" 220 " $4 \times 5 \times 11$ " $2 \times 81 \times 5$
" 275 " 25×11 " 8×81

and so on; that is, in each case, by the remaining factors of the L. C. M.

(20). *Generally* reduce mixed numbers to improper fractions. *Except;* for addition or subtraction, keep them as mixed numbers. Mixed numbers.

(21). Put the symbol of multiplication for the word "of." Compound Fractions.

(22). *Avoid crossing out* so much as to make the work difficult to look over. Rather take two or three extra steps. Simplificatl'n of Fractions.

(23). Invert and multiply. Division of Fractions.

(24). Avoid the clumsy process of bringing the numerator a common denominator and likewise the denominator. Complex Fractions.

But multiply numerator and denominator by such a number as will clear away the compound fractions.

Ex. (1). $$\frac{\frac{1}{6} - 1 + \frac{3}{4}}{3 - \frac{5}{6}} = \frac{4 - 12 + 9}{36 - 10} = \frac{1}{26}$$

If this elementary form of complex fraction is well understood, others of greater complexity will cause no serious

difficulty. Care must always be taken to distinguish the central line of the fraction.

Ex. (2). $\dfrac{\frac{1}{2}+\frac{1}{3}+\frac{1}{4}}{\frac{1}{2\frac{1}{2}}+\frac{1}{3\frac{1}{2}}+\frac{1}{4\frac{1}{2}}} = \dfrac{\frac{6+4+3}{12}}{\frac{2}{5}+\frac{2}{7}+\frac{2}{9}} = \dfrac{13}{12} \times \dfrac{1}{2} \times \dfrac{5\times7\times9}{80+63}$

$$= \frac{5\times7\times3}{8\times11} = \frac{105}{88} = 1\frac{17}{88} \quad \text{Ans.}$$

Note.--For $5\times7+5\times9+7\times9 = 5\times16+63 = 80+63$. Considerations of this kind facilitate mental calculation.

Ex. (3). $\dfrac{1}{11}\times\dfrac{1}{1+\frac{\frac{1}{3}}{3-\frac{1}{4}}} = \dfrac{1}{11}\times\dfrac{1}{1+\frac{4}{36-3}} = \dfrac{1}{11}\times\dfrac{33}{37} = \dfrac{3}{37}$ Ans.

(25). Ex. (1). Reduce $6^{s.}\ 7\frac{6}{7}{}^{d.}$ to the fraction of $7^{s.}\ 9^{d.}$ Fractions of Concrete numbers.

$$6^{s.}\ 7\tfrac{6}{7}{}^{d.} = \frac{6^{s.}\ 7\frac{6}{7}{}^{d.}}{7^{s.}\ 9^{d.}} \times 7^{s.}\ 9^{d.}$$

$\therefore$ The Answer is $\dfrac{6^{s.}\ 7\frac{6}{7}{}^{d.}}{7^{s.}\ 9^{d.}} = \dfrac{79\frac{6}{7}\times 1^{d.}}{93\times 1^{d.}} = \dfrac{559}{651}$ Ans.

These are examples of dividing concrete numbers of like nature to obtain ratios or abstract numbers.

Ex. (2). Reduce 3 roods, $27\frac{1}{2}$ poles to the fraction of an acre.

$$\frac{3^{\text{roods.}}\ 27\frac{1}{2}{}^{\text{poles.}}}{1\text{ acre}} = \frac{147\frac{1}{2}}{160} = \frac{29\frac{1}{2}}{32} = \frac{59}{64}$$

Ex. Reduce 1 oz. Troy to the fraction of 1 oz. Avoirdupois.

$$\frac{1^{\text{oz.}}\text{ Ty.}}{1^{\text{oz.}}\text{ Av.}} = \frac{480^{\text{grs.}}}{\frac{7000^{\text{grs.}}}{16}} = \frac{16\times48}{700} = \frac{192}{175} \text{ Ans.}$$

This is the same thing as saying that an ounce Troy and an ounce Avoirdupois are in the ratio 192 : 175.

(26). The rules for decimals follow from the rules for vulgar fractions, of which they are only a particular case. **Decimal Fractions.**

(27). Only fractions having the same denominator can be at once added or subtracted. **Addition or Subtraction.**

Therefore place the decimal points in a vertical row.

For there are then the same number of 10^s in the denominators.

(28). In multiplying fractions we multiply the numerators for a new numerator, and the denominators for a new denominator. **Multiplicat'n**

In decimals the first is done by the actual multiplication, the second by pointing off as many places of decimals as are in the multiplier and multiplicand together.

(29). Here also the division of the numerators gives the actual figures; the subtraction of the number of decimal places in the divisor from the number in the dividend gives the number in the quotient. **Division.**

Or, it may be easier as follows:

State the sum ready for dividing.

Mark off from the dividend as many decimal places as there are in the divisor, (adding cyphers if necessary).

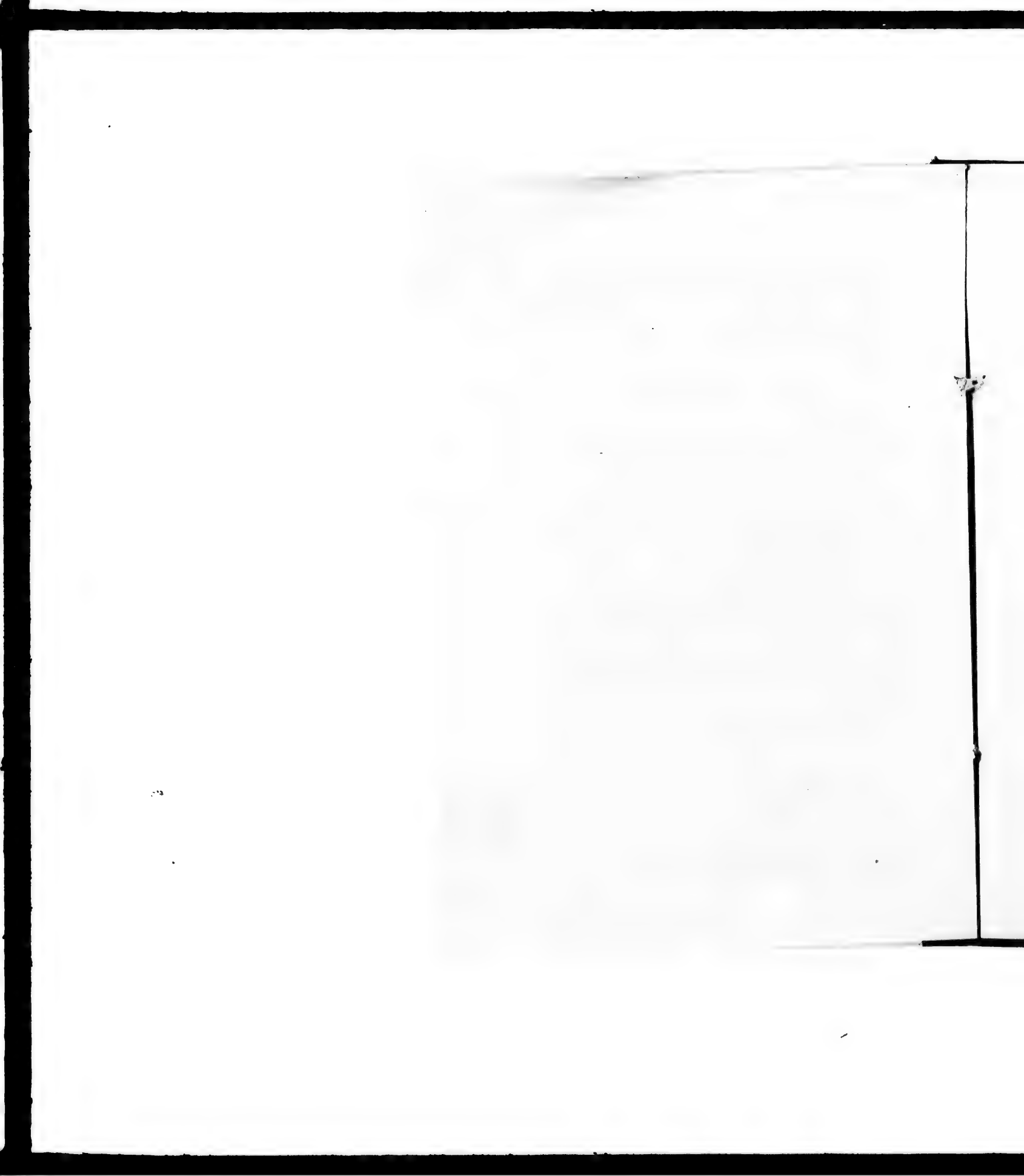
Divide up to this point, then put the decimal point, and continue if necessary.

This is practically the same as multiplying both divisor and dividend by a certain number of 10^s, which does not affect the result.

Ex. (1). ·37)·00|01813(·00049 Ans.
148
333

Ex. (2). ·037)181·300(4900 Ans.
148
333

These two examples include all possible cases.



(30). $\frac{1}{2}=\cdot5$; $\frac{1}{4}=\cdot25$; $\frac{1}{8}=\cdot125$; $\frac{1}{16}=\cdot0625$

$\frac{1}{3}=\cdot\dot{3}$; $\frac{1}{6}=\cdot1\dot{6}$; $\frac{1}{9}=\cdot\dot{1}$; $\frac{1}{12}=\cdot08\dot{3}$

$\frac{1}{5}=\cdot2$; $\frac{1}{25}=\cdot04$; $\frac{1}{125}=\cdot008$; $\frac{1}{625}=\cdot0016$

$\frac{1}{7}=\cdot\dot{1}42857$

$\frac{2}{7}=\cdot2\dot{8}5714$

$\frac{3}{7}=\cdot4285\dot{7}1$

$\frac{4}{7}=\cdot571\dot{4}2\dot{8}$

$\frac{5}{7}=\cdot714\dot{2}\dot{8}\dot{5}$

$\frac{6}{7}=\cdot8571\dot{4}\dot{2}$

$\frac{7}{7}=\cdot\dot{9}\dot{9}9999=1$

$\frac{1}{11}=\cdot0\dot{9}$; $\frac{2}{11}=\cdot1\dot{8}$; $\frac{3}{11}=\cdot2\dot{7}$; $\frac{9}{11}=\cdot8\dot{1}$; $\frac{10}{11}=\cdot\dot{9}0$

Simple Fractions expressed as Decimals.

The above decimals should all be carefully examined, as it frequently saves much laborious work, if the source of a circulating decimal is recognized.

It must be observed that all vulgar fractions can be expressed as decimals which will terminate, if the denominator contains no other prime factor than 2 and 5, but which will otherwise recur.

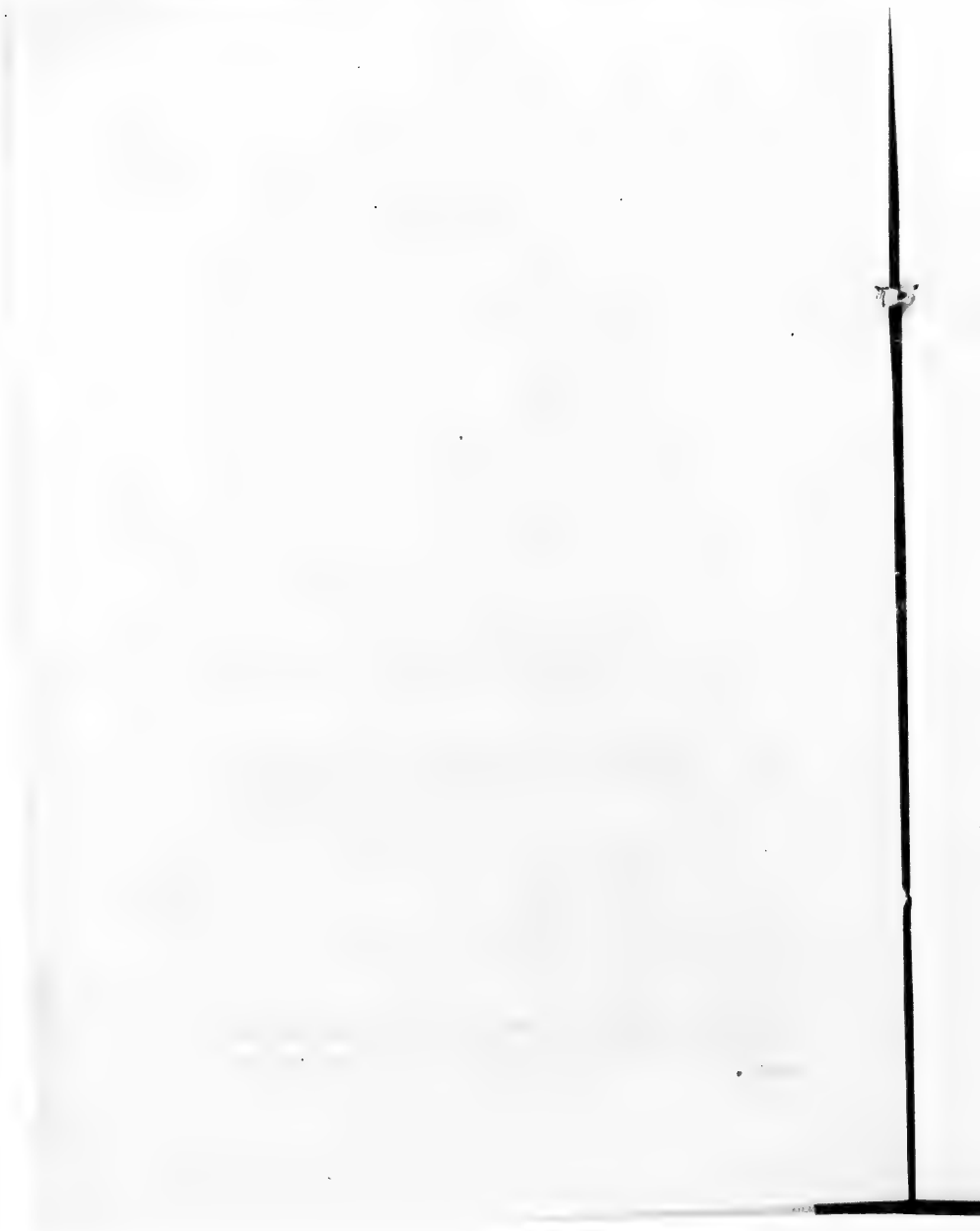
(31). It is most important to observe that $\cdot\dot{9}=1$.

Hence it is plain that $\cdot239\dot{9}=\cdot24$.

Circulating Decimals.

Observe that when a single figure recurs it is due to a denominator 3 or 9.

When two figures recur and their sum equals 9, such as $\cdot0\dot{9}$; $\cdot2\dot{7}$; $\cdot\dot{9}0$, &c., &c., division by 11 has been the cause.



Ex. Reduce $25{\cdot}3\dot{5}\dot{4}$ to a vulgar fraction.

Multiply numerator and denominator by 11.

$$\therefore 25{\cdot}3\dot{5}\dot{4} = \frac{278{\cdot}9}{11} = \frac{2789}{110}$$

$$\text{or} = 25\frac{3{\cdot}9}{11} = 25\frac{39}{110}$$

$$\text{or it may be written as } 25{\cdot}3\frac{6}{11}$$

(32). The number 7 deserves particular notice, for when $\frac{1}{7}$, $\frac{2}{7}$, $\frac{3}{7}$, &c., are expressed as decimals, the same six figures are always involved in the circulating portion, and always in the same order. These figures should always be recognized when met with.

The No. $142857 \times 7 = 999999$.

This No. is composed of 14, 28, 57; 28 being twice 14, and 57 being 1 more than twice 28 (the 1 being in reality carried from 112, the double of 56).

Thus it should be easily remembered.

Ex. Reduce $58{\cdot}3\dot{9}4285\dot{7}1 = 58{\cdot}39\frac{3}{7} = \frac{40876}{700}$ or $= 58\frac{276}{700}$

This is a much simpler result than could be directly obtained by any general method.

(33). Let N be the circulating decimal.

Recurring or Circulating Decimals expressed as Fractions.

Multiply N by as many 10^s as there are decimal places to the end of the first recurring section; and again by as many 10^s as there are decimal figures to the beginning of the same section; (i.e. in following Ex. by 10^5 & 10^2) and subtract. Thus the circulating part vanishes.

Ex. Let $N = 8{\cdot}02\dot{0}8\dot{3}$

$$\therefore 10^5 \times N = 100000\,N = 802083{\cdot}\dot{0}8\dot{3}$$

$$10^2 \times N = \quad 100\,N = \quad 802{\cdot}\dot{0}8\dot{3}$$

$$\therefore 99900\,N = 801281 \quad \therefore N = \frac{801281}{99900}$$

Or, setting aside the whole number, $N = 8\frac{2081}{99900}$

Hence the Rule.

Recurring Decimals.

Subtract the non-recurring portion (including integer if any) *from the whole and divide by as many* 9s *as there are recurring figures, followed by as many cyphers as non-recurring places of decimals.*

Thus the above example $58\cdot39428571\dot{}$ becomes

$$\frac{5839422732}{99999900} \text{ or } 58\frac{39428532}{99999900}$$

These would be found, after much trouble, to reduce to $\frac{10219}{175}$ or $58\frac{69}{700}$ obtained so easily above.

Addition or Subtraction.

(34). Rule. Place the decimals in order for addition. Mark off by a vertical line all non-recurring figures.

Complete each recurring section to the right of the vertical line.

Continue each recurring decimal by a complete section at a time until all terminate at a second vertical line.

Add, taking care to carry the correct number; and the result between the vertical lines will recur.

Ex. (1). Add $\left\{\begin{matrix}2\cdot478\dot{}\\1\cdot226\dot{}\end{matrix}\right\}$ i.e. $\left\{\begin{matrix}2\cdot4|784\dot{}\\1\cdot2|26\dot{}\end{matrix}\right\}$ i.e. $\left\{\begin{matrix}2\cdot4|784784\dot{}\\1\cdot2|262626\dot{}\end{matrix}\right\}$

$3\cdot7047411\dot{}$ Ans.

Ex. (2.) Subtract the same numbers $1\cdot2\dot{5}22158\dot{}$.

It may be observed that the number of recurring figures in the answer is the L. C. M. of the numbers of recurring figures in the separate decimals.

Multiplicat'n and Division.

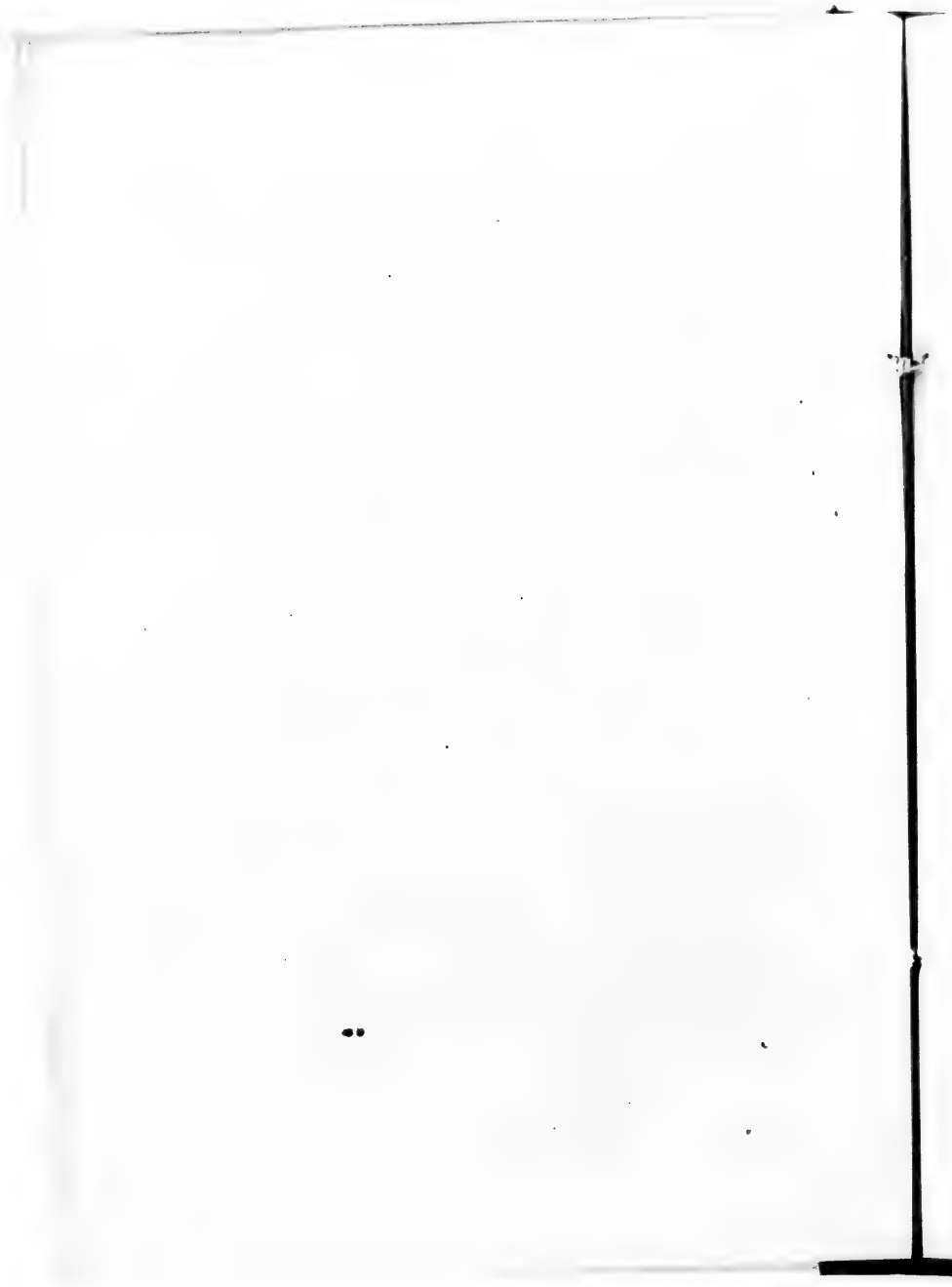
(35). Recurring decimals may be multiplied or divided, care being taken to make the proper carryings.

Never multiply or divide by a recurring decimal; it is most dangerous as well as cumbrous.

In such cases transform, if necessary, to a vulgar fraction.

Ex. Multiply $3\cdot4\dot{3}7\dot{}$ by $1\cdot571428\dot{}$ i.e. by $1\frac{4}{7}$

$$\begin{array}{r} 3\cdot4\dot{3}7\dot{} \\ 11 \\ \hline 7)37\cdot811\dot{} \\ \hline 5\cdot4015873\dot{} \end{array}$$ Ans.



(36). The last figure of a non-terminating decimal must be taken as near as possible to the correct result, i.e. 1 must be added to it if followed by more than 5.

Ex. π written to 5 places is 3·14159
" 4 " 3·1416
" 3 " 3·142
" 2 " 3·14

(37). See Fractions, page 13. The process is the same, but there is a shorter method. Decimal of a Concrete number.

Ex. Reduce 4s. 7½d. to the decimal of a pound.

s. d.
4 7·5
12)
20)4·625 Shillings.
£·23125 Ans.

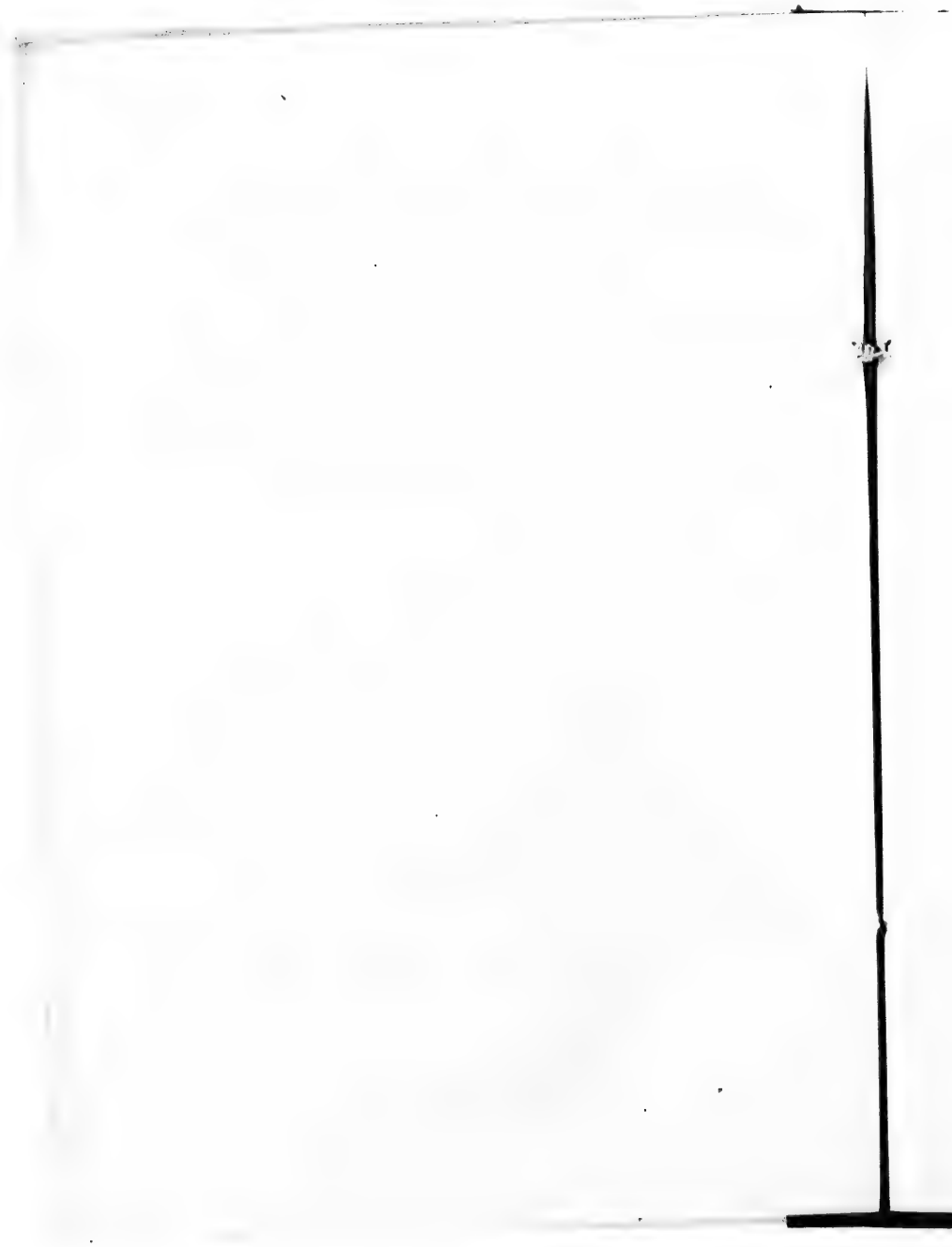
Ex. (2). Reduce 11 seconds to the decimal of 5 days.

60) 11 seconds.
60) ·18̇3̇ minutes.
12)·00305̇ hours.
2)·000254629̇
5)·000127314814̇ days.
·0000254629̇ Ans.

Of course the last two divisions could be done more easily in one.

Ex. (2). Reduce 6 furlongs, 41 yds. to the decimal of a mile.

6 furlongs 41 yards.
20)
11)2·05
8)6·1863̇ furlongs.
·7732954̇ of a mile. Ans.



(38). There are three type cases of sums to be considered. Practice.

First. A compound concrete number multiplied by a large abstract number. Ex. £2 : 13 : $3\frac{1}{2} \times 239$. See Art. 15.

		239			239
		2			2
s.	£	478		£	478
10 is	$\frac{1}{2}$	119 : 10	13s 4d = $\frac{1}{3} \times 2$		159 : 6 : 8
2s. : 6d.	$\frac{1}{4}$	29 : 17 : 6	$-239 \times \frac{1}{2}$d =		$-9 : 11\frac{1}{2}$
6d.	$\frac{1}{8}$	5 : 19 : 6			$636 : 16 : 8\frac{1}{2}$
3d.	$\frac{1}{16}$	2 : 19 : 9			Ans.
$\frac{1}{2}$d.	$\frac{1}{96}$	$9 : 11\frac{1}{2}$			
		$636 : 16 : 8\frac{1}{2}$ Ans.			

Second. Find the value of 9yds. 2ft. 10in. at 5s. $7\frac{1}{2}$d. per yard. That is, multiply 9yds. 2ft. 10in. $\times \frac{5\text{s. : } 7\frac{1}{2}\text{d.}}{1\text{yd.}}$

The unit of length divides out ∴ the Answer is in £. s. d. viz.: 5s. $7\frac{1}{2}$d. $\times \left(9\frac{2}{3} + \frac{10}{36}\right)$

Place the money at the head of the column and break up the quantity into aliquot parts.

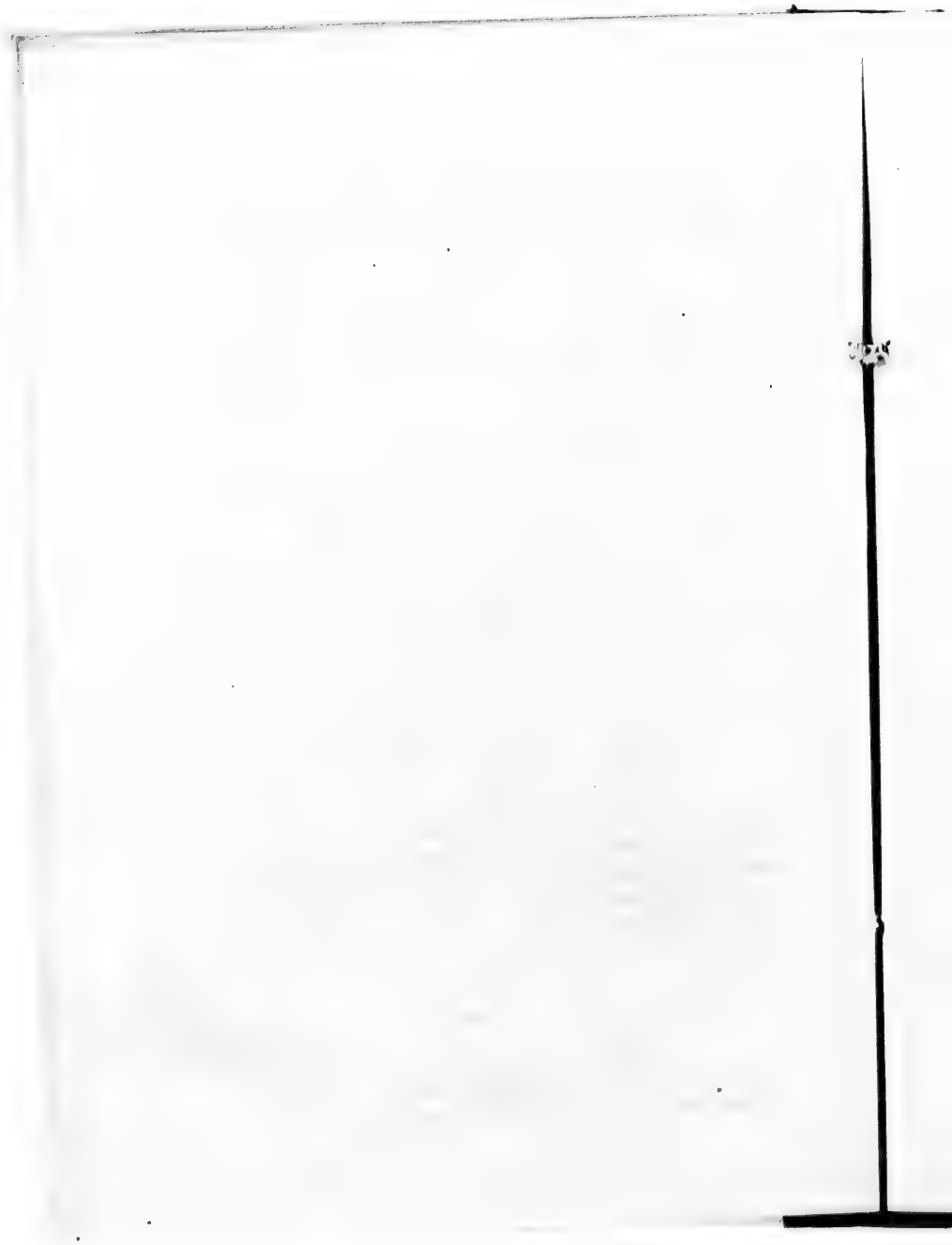
	s.	d.
	5 :	$7\frac{1}{2}$
		9
	2 : 10 :	$7\frac{1}{2}$ = value of 9yds.
1ft. 6in. = $\frac{1}{2}$yd.	2 :	$9\frac{3}{4}$ = " 1ft. 6in.
1ft. = $\frac{1}{3}$yd.	1 :	$10\frac{1}{2}$
4in. = $\frac{1}{3}$ft.		$7\frac{1}{2}$
	£2 : 15 :	$11\frac{1}{4}$ Ans.

Third. Calculations of the following kind frequently occur in logarithmic work, and "practice" is far the most convenient method.

If ·0003567 corresponds to 1 minute, what corresponds to 17 seconds.

	·0003567
$15^{\text{sec.}} = \frac{1}{4}$min.	8917
$2^{\text{sec.}} = \frac{1}{30}$min.	1189
	·0001011 Ans.

Care must be taken not to add in the top line.



Ex. Find log. sin. 19° 10′ 37″. Diff. = 3633

log. sin. 19° 10′	= 9·5162936
30″	18165
6″	3633
1′	605
∴ log. sin. 19° 10′ 37″	= 9·5165176

This is specially useful when several logarithms have to be added together.

Ex. Add log. 296 + log. sin. 32° 18′ 53″ + log. sec. 54° 41′ 28″

log. 296	= 2·4712917		
log. sin. 32° 18′	= 9·7278277		D = 1998
30	999		
20	666		
3	99	9	
log. sec. 54° 41′	= 10·2380008		D = 1784
20	594	7	
6	178	4	
2	59	5	
∴ required log.	= 22·4373799		

(39). State the sum (without calculation if possible) in the form of proportion, so as to agree with the wording. Rules for Rule of Three

The third term must be of the same nature as the answer.

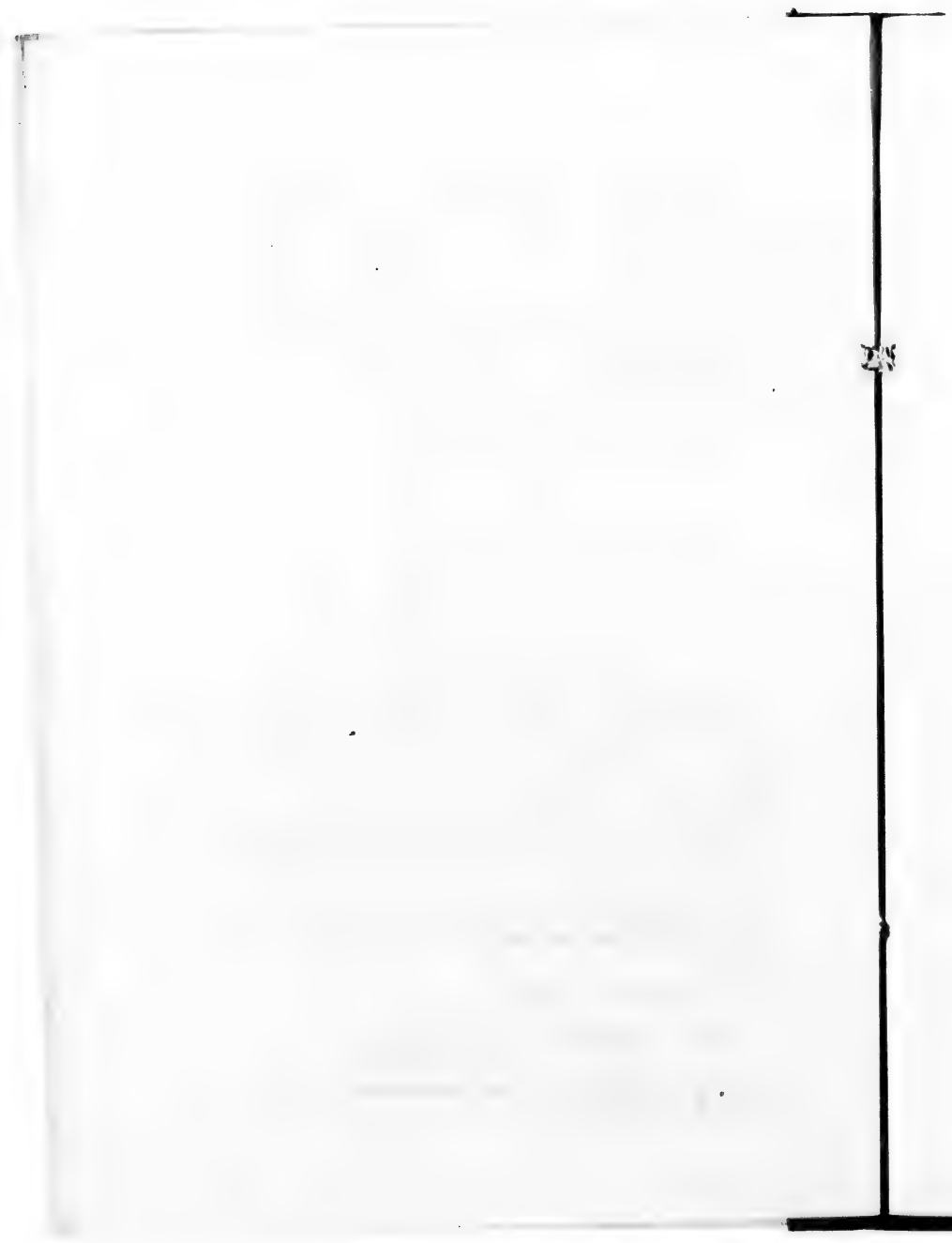
The first and second terms must be of the same nature, and when brought to the same denomination, may be treated as abstract numbers, because the concrete unit divides out.

Consider whether the answer should be greater or less than the third term, and arrange the first two terms accordingly.

greater : less : : greater : { Ans. if less. }

less : greater : : less : { Ans. if greater. }

Reduce by dividing out common factors from the 1st and 2nd or 1st and 3rd terms.



Multiply 2nd and 3rd and divide by 1st.

Ex. If 10 men can mow a field in 12 days, in how many days will 15 men mow it? Evidently in less days.

men.	men.	days.
15 :	10 : :	12
3	2	
1	2	4
		2
		8 days. Ans.

Double Rule of Three.

(40). When the result depends on several independent proportions, consider each proportion separately; treating the others as if they were constant.

Multiply all the first terms for a new first term, and all the second terms for a new second term, and proceed as for Simple Rule of Three.

The concrete units divide out from all the first and second terms as before.

Ex. If it cost $84 to keep 3 horses for 7 months, what will it cost to keep 2 horses for 11 months?

		$	
horses	3 : 2 : :	84	: less for fewer horses.
months	7 : 11		: more for longer time.
	21 : 22 : :	84	
		4	
		$88	Ans.

Interest, Discount, Present worth

(41). Interest, Present Worth and Discount are all to be worked by Rule of Three.

If $2 is the interest of $100 for any length of time, then $2 is the discount of $102 for the same time, and $100 is the present worth of $102.

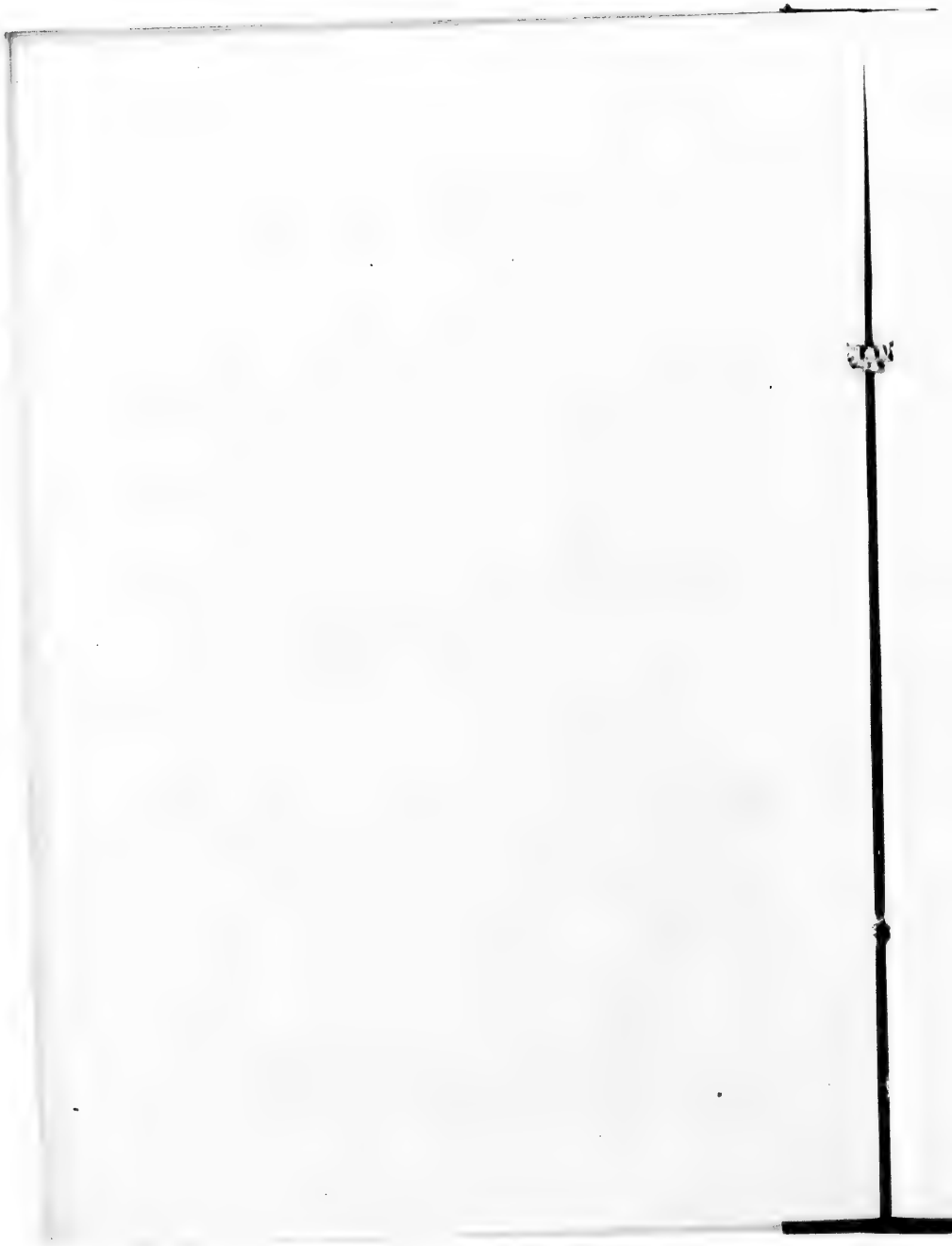
If this is clearly understood there is no difficulty in working examples.

The statements are as follows: Ex. 2 per cent for 3 mos.

100 : { its interest / 2 } : : The sum : its *Interest.*

102 : { its discount / 2 } : : The sum : its *Discount.*

102 : { its present worth / 100 } : : The sum : its *Present Worth*



Interest, Discount, Present worth

The interest of $100 for any fixed time and rate is the time multiplied by the rate. Hence the rules.

For Interest.—Multiply by the rate and time and divide by 100.

For Discount.—Multiply as before and divide by (100 + the multiplier).

For Present worth.—Same division, but multiply by 100.

The above rule for interest is better, for more reasons than one, than the method usually taught on the American Continent, which is only applicable to a Decimal coinage. The above rule is applicable to every coinage, and is frequently shorter even for a Decimal coinage.

Ex. (1). Interest on $556.50 for $1\frac{1}{4}$ years at 8 per cent.

$$\text{Here } \frac{\frac{5}{4} \times 8}{100} = \frac{10}{100} = \frac{1}{10}$$

$\therefore$ Interest = 55.65 Ans.

Ex. (2). Interest on £1534 6s 3d for $1\frac{3}{8}$ years at 8 per cent.

$$\begin{array}{r|l} & 11 \\ 168 & 78\ 8\ 9 \\ 15 & 48 \\ 5 & 85 \end{array}$$

$\therefore$ Interest = £168 15 s. 5·85 d.

$= £168\ 15\ 5\frac{17}{20}$ Ans.

Ex. (3). Discount on $487 due 5 mos. hence at 7 per cent.

$$\text{Interest on \$100} = \frac{5 \times 7}{12} = \frac{35}{12} = \text{Discount on } 100 + \frac{35}{12}$$

$$\therefore 100 + \frac{35}{12} : \frac{35}{12} :: 487 : 7$$

$$1235 : 35$$

$$247 : 7$$

$$247)3409(\$13.80\tfrac{40}{247} \quad \text{Ans.}$$

247

939

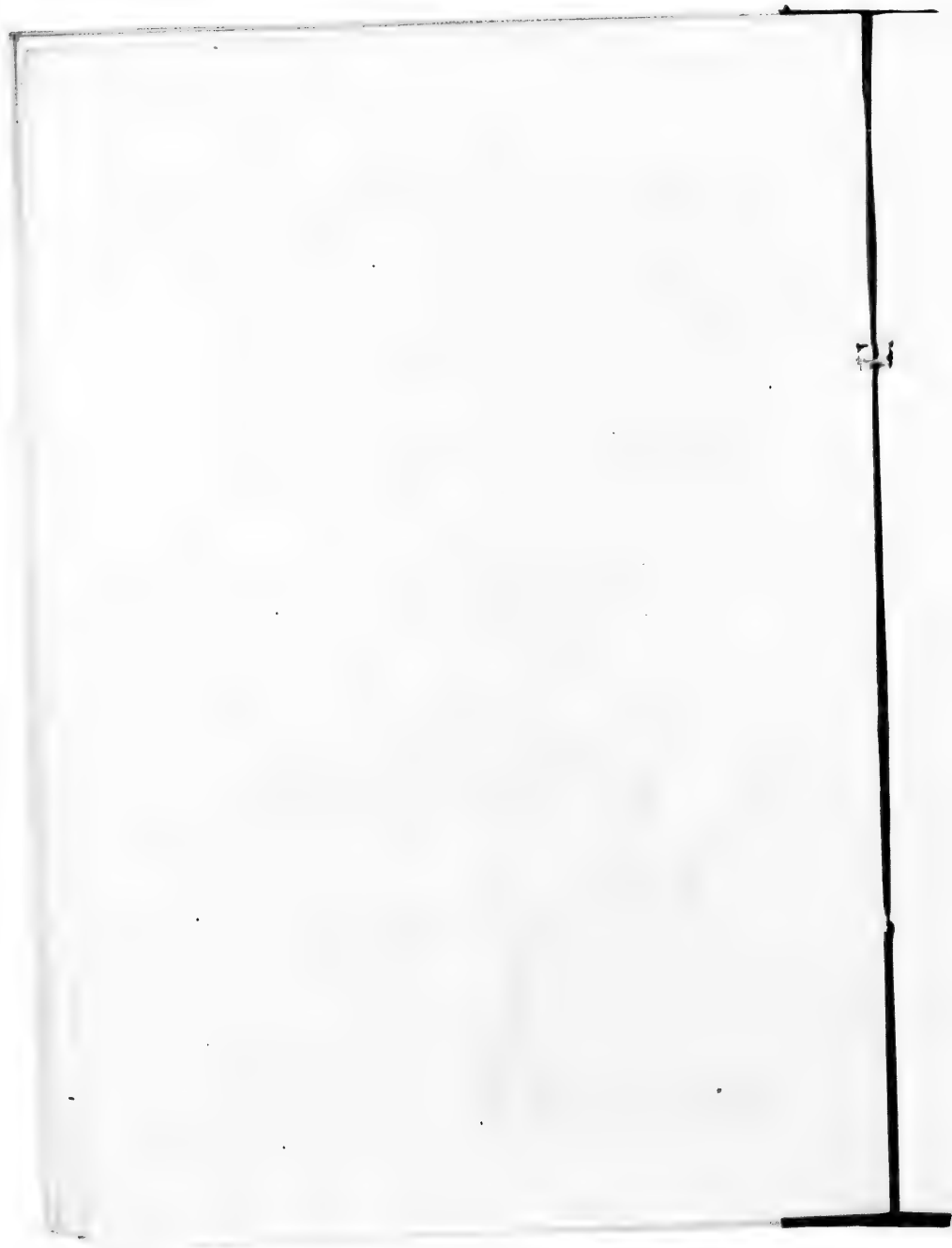
741

1980

1976

40

$\therefore$ The Present Worth = 473.19\frac{207}{247}$



Ex. (4). Present worth of $340 due 5 mos. hence at 6 per cent.

$$100 + \frac{5 \times 6}{12} : 100 : : 340$$

$$205 : 100 : : 680$$

$$41) 13600) \$331.70\tfrac{30}{41} \quad \text{Ans.}$$

123
130
123
70
41
290
287
30 ∴ The discount = $\$8.29\frac{11}{41}$

(42). If a stock is quoted at a sum different from 100. The interest will be found in the same manner, but the divisor will be the sum quoted, instead of 100. Stocks.

Ex. Find the yearly income from $1008 invested in the 6 per cents at 84.

$$\text{Income} = \frac{1008 \times 6}{84} = \$84 \quad \text{Ans.}$$

Brokerage increases the cost of the investment. Brokerage.
diminishes the proceeds of sale.

(43). The following Algebraical formulæ may be advantageously inserted here :— Interest.

Let P = Principal or Present Worth.
I = Interest for whole time.
M = P + I = Amount.
D = M − P = Discount on M.

Hence it is evident that I on P = D on M.

Let r = rate per cent.

$R = 1 + \frac{r}{100}$ = one dollar plus its interest for 1 year.

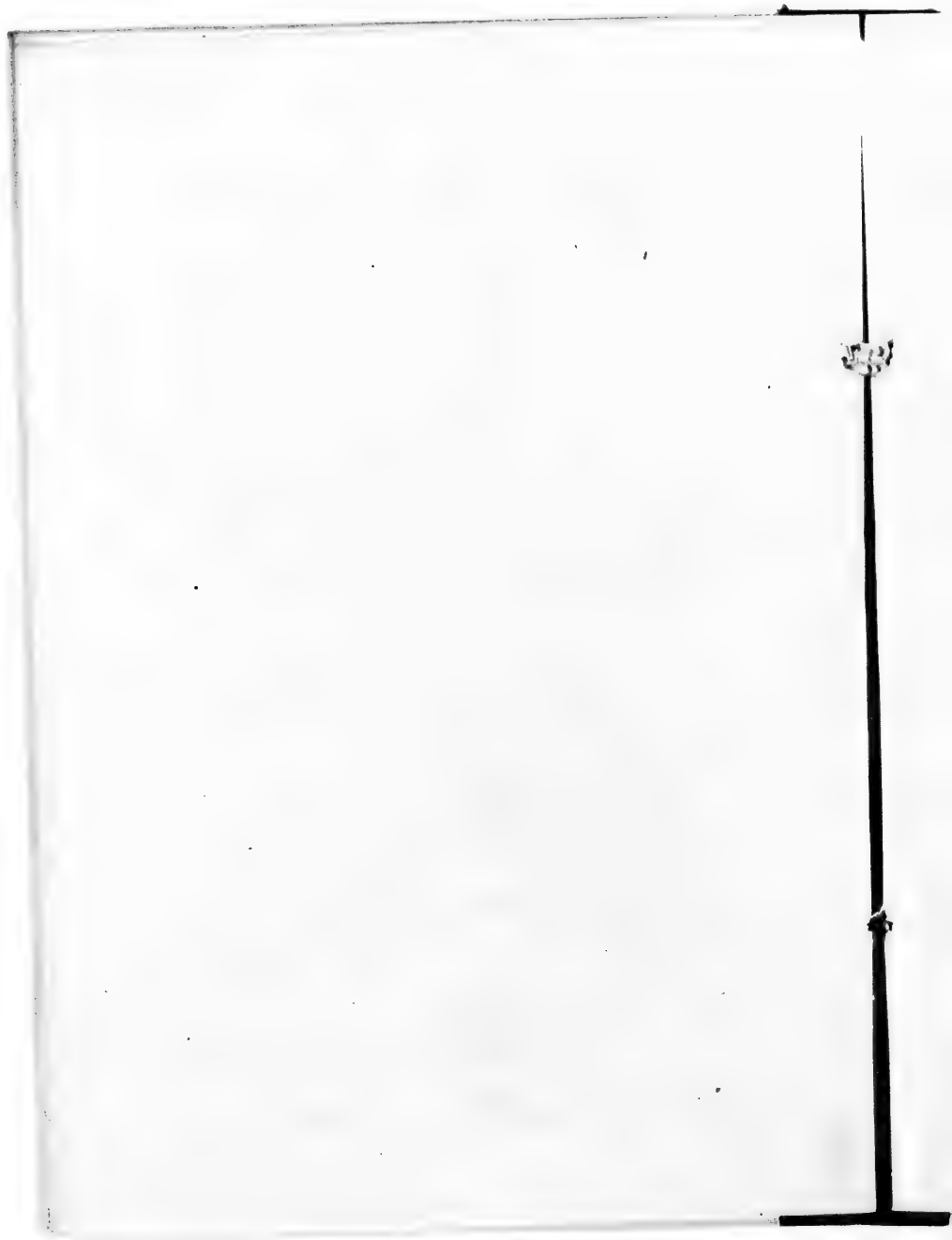
Let n = time in years.

$\therefore I = \frac{Prn}{100}$ Hence the Rule. Interest.

$\therefore M = P + I = P\left\{1 + \frac{rn}{100}\right\}$

$\therefore P = \frac{M}{1 + \frac{rn}{100}} = \frac{100\,M}{100 + rn}$ Hence the Rule. Present worth

$\therefore D = M - P = \frac{Mrn}{100 + rn}$ Hence the Rule. Discount.



(44). All interest and discount ought, in justice, to be calculated at compound interest. Compound Interest.

Let M_1, M_2, &c., M_n mean the amount at the end of 1, 2, &c., n years respectively.

$\therefore M_1 = P\left\{1 + \frac{r}{100}\right\} = P\,R$ = new principal.

$\therefore M_2 = PR\left\{1 + \frac{r}{100}\right\} = P\,R \times R = P\,R^2$

$\therefore M_n = P\,R^n$

Questions in compound interest should generally be calculated as logarithms.

To find present worth and discount:

$$P = \frac{M}{R^n}$$

$$\therefore D = M - P = M.\frac{R^n - 1}{R^n} = M\left\{1 - \frac{1}{R^n}\right\}$$

(45). Let P be the annuity. Present Value of an Annuity calculated at Compound Interest.

$\therefore$ Amount at end of n years $= P\left\{R^{n-1} + R^{n-2} + \ldots + 1\right\}$

$$= P \times \frac{R^n - 1}{R - 1} \qquad \text{(Prove by dividing out).}$$

$$\therefore \text{Present Value} = \frac{M}{R^n} = P.\frac{R^n - 1}{(R-1)R^n}$$

$$\therefore \text{Present Value of a perpetual annuity} = P.\frac{1 - \frac{1}{R^n}}{R - 1}$$

$$= \frac{P}{R-1} = \frac{100\,P}{r}$$

i.e = the sum whose interest is the annuity.

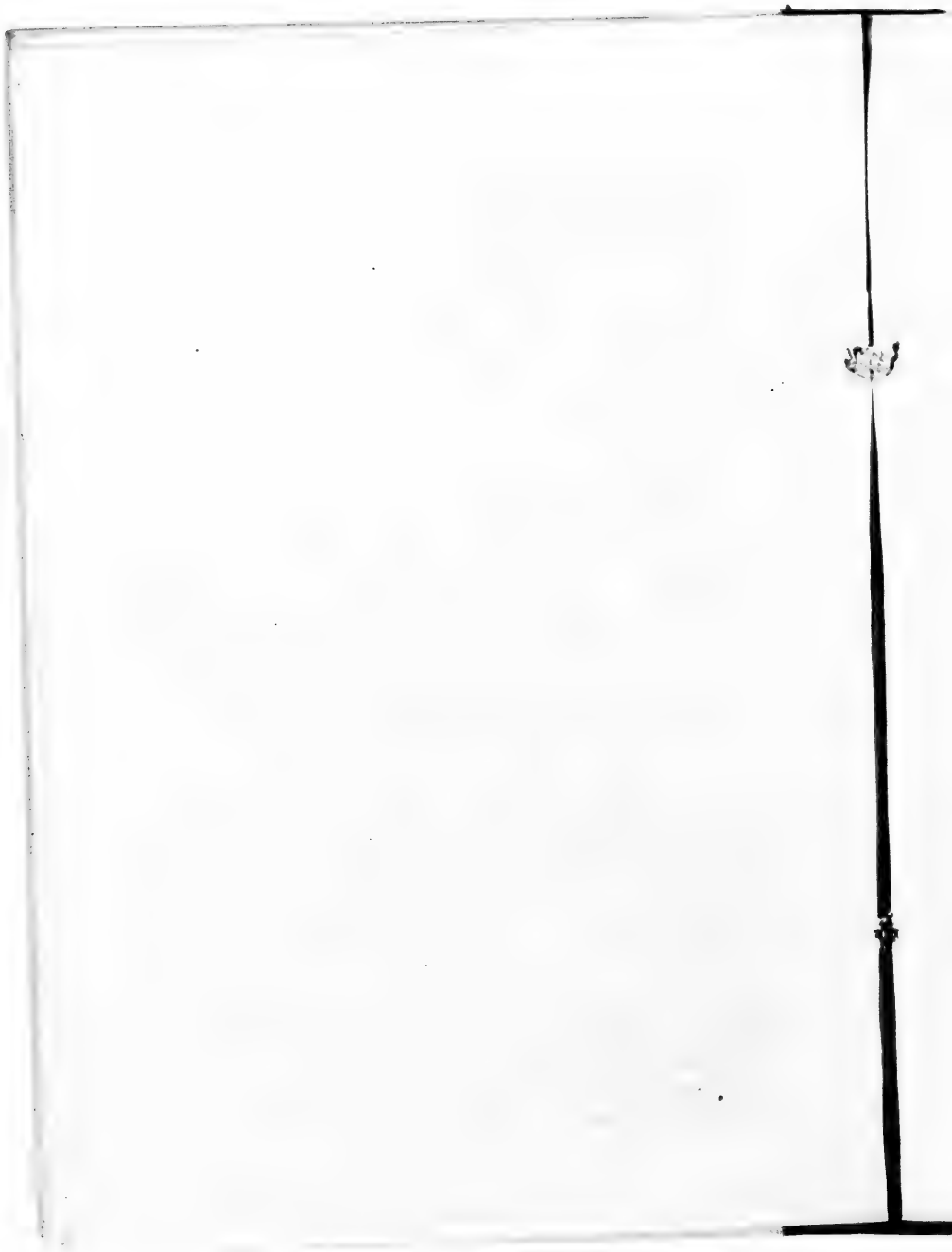
(46). These are generally examples of Rule of Three, 100 being the 2nd or 3rd term of the proportion. Per Cent.

(47). Rule. As the sum of the given parts is to any one of them, so is the entire quantity to be divided to the corresponding part of it. Proportional Parts.

See examples in any book on Arithmetic.

(48). It will be observed that the method of pointing adopted on the American continent differs from the old method in use in England. Square Root.

There is no apparent advantage gained by the alteration, but it introduces confusion into the minds of those who meet with both systems.



The learner must bear in mind that the dots merely serve to mark off sections of two figures from the right of the integral portion towards the left, and again from the decimal portion towards the right. Square Root.

Ex. 13|47|28·59|63|2.

There will be as many figures in the Answer as there are sections.

The object of the method is to find the nearest square number successively below 13; 1347; 134728; 134738·59, &c., and depends upon the Algebraical formula.

$(a+b)^2 = a^2 + 2ab + b^2 = a^2 + b(2a+b)$

where a stands for the root already found, and b for the next figure.

$a^2 = 900$ | $2a+b$ | $\therefore 2ab+b^2 =$

```
                13|47|28·59,63|2  (367·054.  Ans.
                 9
66  )447
      396
727 )5128
      5089
73405 )395963
       367025
        28938
```

An additional number of figures, one less than the number already found, can be obtained by common division. That is, in the above Example a total of seven figures could be obtained by dividing 3959632 by 7340. See Todhunter's large Algebra.

(49). The above remarks, with reference to pointing, apply also to cube root, but the sections comprise 3 figures each. Cube Root.

Rule. Mark off every three figures both ways from the decimal point.

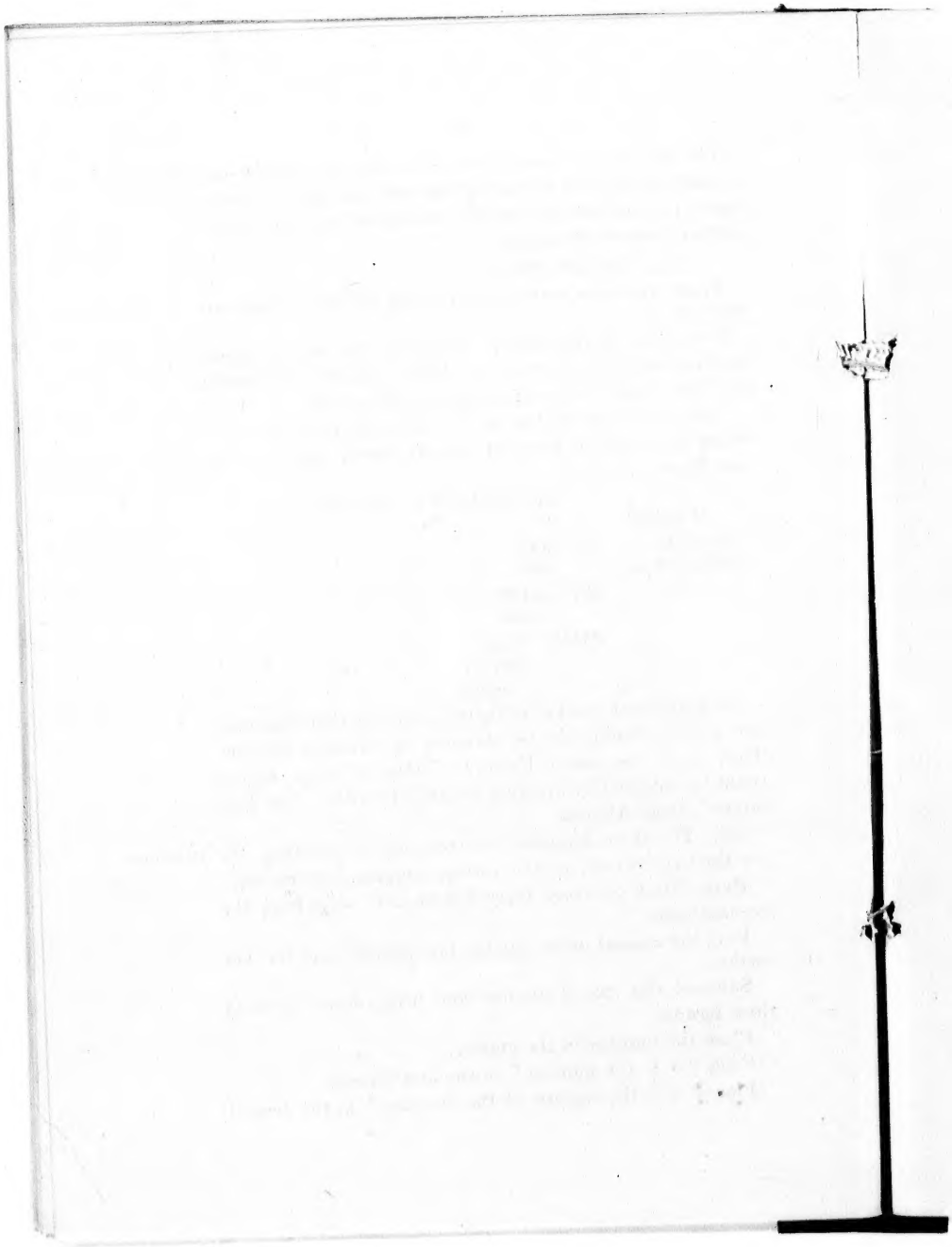
Find the nearest cube number not greater than the left section.

Subtract this cubed number and bring down the next three figures.

Place the number in the answer.

Place "3 × the number" in the first column.

Place "3 × the square of the number" in the second



column and place two cyphers on the right. This is the trial divisor, which is not very reliable till after two or three figures of the answer have been obtained.

Place in the answer the probable quotient.

Place this number in the 1st column, on the right.

Add in to the 2nd column the product of the number and the 1st column.

This is the real divisor.

Divide and bring down the next three figures.

To obtain a third figure we get a new first column by trebling the last figure (i.e. by trebling the whole answer already found).

We get a new 2nd column by adding in the square of the last figure found to the two last rows in the 2nd column, and placing two cyphers on the right as before (i.e. by trebling the square of the whole answer already found).

The proof depends on the Algebraical formula.

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$= a^3 + b\ (3a^2 + b\ \overline{3a+b}).$$

Whenever a cypher is found in the answer, place one cypher on to the 1st column, two cyphers to the 2nd column and bring down the next three figures.

Ex. 184 = $3a+b$	10800 = $3a^2$	264|609|288(642	Ans.
	736 = $b(3a+b)$	216	
	11536 real divisor.	48609	
	16	46144	
1922	1228800 = $3(a+b)^2$	2465288	
	3844		
	1232644	2465288	
Ex. 2101	1470000	344·5(7·0102 nearly.	Ans.
	2101	343	
	1472101	1500000	
	1	1472101	
21030	14742030000	27899000000	

Additional figures, two less than those already found, can be obtained by common division. Thus a total of 6 figures can be obtained in the last example by using the last real divisor. See Todhunter's Algebra.